APPENDIX B

EULER EQUATIONS

1. Variational Principle in Transformed Space

Consider the integral

$$I = \int F[\underline{g}, w(\underline{x})] d\underline{\xi}$$

where $\underset{\sim}{g}$ is the covariant metric tensor, with elements g_{ij} defined by Eq. (III-5), and $w(\underset{\sim}{x})$ is a weight function dependent on $\underset{\sim}{x}$.

A. Grid Generation System

The Euler equations then are given by

$$\sum_{j=1}^{3} \frac{\partial}{\partial \xi^j} \frac{\partial F}{\partial (x_i)_{\xi^j}} - \frac{\partial F}{\partial x_i} = 0 \qquad (i = 1,2,3) \tag{2}$$

as has been noted. Since

$$(x_i)_{\xi^j} = (a_j)_i$$

and F depends on $(x_i)_{\xi^j}$ only through the elements of the metric tensor, $\underset{\sim}{g}$, we have

$$\frac{\partial F}{\partial (x_i)_{\xi^j}} = \frac{\partial F}{\partial (\underline{a}_j)_i} = \frac{\partial F}{\partial \underline{a}_j} \frac{\partial \underline{a}_j}{\partial (\underline{a}_j)_i} = \frac{\partial F}{\partial \underline{a}_j} \underline{e}_i \tag{3}$$

where $\underset{\sim}{e}_i$ is the unit vector in the x_i-direction. Here the operation indicated by the notation, $\partial F/\partial \underset{\sim}{a}_j \; \underset{\sim}{e}_i$, is the simple replacement of $\underset{\sim}{a}_j$ by $\underset{\sim}{e}_i$ in F. Also, since F depends on $\underset{\sim}{a}_j$ only through $\underset{\sim}{g}$, we have

$$\frac{\partial F}{\partial \underline{a}_j} \underline{e}_i = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g_{kl}} \frac{\partial g_{kl}}{\partial \underline{a}_j} \underline{e}_i$$

$$= \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g_{kl}} \frac{\partial (\underline{a}_k \cdot \underline{a}_l)}{\partial \underline{a}_j} \underline{e}_i$$

or

$$\frac{\partial F}{\partial \underline{a}_j} \underline{e}_i = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g_{kl}} [\delta_{kj}(\underline{a}_l \cdot \underline{e}_i) + \delta_{lj}(\underline{a}_k \cdot \underline{e}_i)]$$

$$= \sum_{l=1}^{3} \frac{\partial F}{\partial g_{jl}} (\underline{a}_l \cdot \underline{e}_i) + \sum_{k=1}^{3} \frac{\partial F}{\partial g_{kj}} (\underline{a}_k \cdot \underline{e}_i)$$

Therefore,

$$\frac{\partial F}{\partial (x_i)_{\xi^j}} = \sum_{k=1}^{3} (\frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}})(\underline{a}_k \cdot \underline{e}_i)$$

Since F depends on $\underline{x}$ only through the weight function we have

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial x_i} = \frac{\partial F}{\partial w} (\underline{\nabla} w)_i = \frac{\partial F}{\partial w} \underline{e}_i \cdot \underline{\nabla} w$$

Then the Euler Equations can be written as

$$\{ \sum_{j=1}^{3} \sum_{k=1}^{3} [(\frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}})\underline{a}_k]_{\xi^j} - \frac{\partial F}{\partial w} \underline{\nabla} w\} \cdot \underline{e}_i = 0 \qquad (i = 1,2,3)$$

or as the vector equation

$$\sum_{j=1}^{3} \sum_{k=1}^{3} [(\frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}})\underline{a}_k]_{\xi^j} - \frac{\partial F}{\partial w} \underline{\nabla} w = 0 \qquad (6)$$

(Note that the symmetric elements of the metric tensor, $g_{jk} = g_{kj}$, are to be left as distinct elements in F until after the differentiation has been performed.)

Expanding the ξ^j-derivative, we then have

$$\sum_{j=1}^{3} \sum_{k=1}^{3} [(\frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}})\underline{r}_{\xi^j \xi^k} + (\frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}})_{\xi^j} \underline{r}_{\xi^k}]$$

$$- \frac{\partial F}{\partial w} \underline{\nabla} w = 0$$

But also

$$\left(\frac{\partial F}{\partial g_{kj}}\right)_{\xi^j} = \sum_{m=1}^{3}\sum_{n=1}^{3}\left[\frac{\partial}{\partial g_{mn}}\left(\frac{\partial F}{\partial g_{kj}}\right)\right](g_{mn})_{\xi^j}$$

$$+ \left[\frac{\partial}{\partial g_{kj}}\left(\frac{\partial F}{\partial w}\right)\right] \underset{\sim}{\nabla} w \cdot \underset{\sim}{r}_{\xi^j}$$

so that

$$\left(\frac{\partial F}{\partial g_{kj}}\right)_{\xi^j} = \sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 F}{\partial g_{mn}\partial g_{kj}}\left(\underset{\sim}{r}_{\xi^m}\cdot\underset{\sim}{r}_{\xi^j\xi^n} + \underset{\sim}{r}_{\xi^n}\cdot\underset{\sim}{r}_{\xi^j\xi^m}\right)$$

$$+ \left[\frac{\partial}{\partial g_{kj}}\left(\frac{\partial F}{\partial w}\right)\right]\left(\underset{\sim}{\nabla} w \cdot \underset{\sim}{r}_{\xi^j}\right)$$

Thus we have the grid generation system, with $\partial F/\partial w$ written as F',

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\Big[A_{jk}\ \underset{\sim}{r}_{\xi^j\xi^k} + A'_{jk}\left(\underset{\sim}{\nabla} w\cdot\underset{\sim}{r}_{\xi^j}\right)\underset{\sim}{r}_{\xi^k}$$

$$+ \sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial A_{jk}}{\partial g_{mn}}\left(\underset{\sim}{r}_{\xi^m}\cdot\underset{\sim}{r}_{\xi^j\xi^n}\right. \tag{7}$$

$$\left.+ \underset{\sim}{r}_{\xi^n}\cdot\underset{\sim}{r}_{\xi^j\xi^m}\right)\underset{\sim}{r}_{\xi^k}\Big] - F'\ \underset{\sim}{\nabla} w = 0$$

where

$$A_{jk} = \frac{\partial F}{\partial g_{jk}} + \frac{\partial F}{\partial g_{kj}} \tag{8}$$

This is a quasi-linear, second-order partial differential equation for the cartesian coordinates $\underset{\sim}{r}$.

If the weight function depends directly on ξ, instead of on $\underset{\sim}{x}$ in Eq. (1), then $\partial F/\partial x_i = 0$ in Eq. (2). Also in his case, the $\underset{\sim}{\nabla} w \cdot \underset{\sim}{r}_{\xi^j}$ that appears on p. 439 and in the development that leads to Eq. (7) is replaced by simply w_{ξ^j}. Then Eq. (7) is replaced by

$$\sum_{j=1}^{3} \sum_{k=1}^{3} [A_{jk} \underline{r}_{\xi^j \xi^k} + A'_{jk} w_{\xi^j} \underline{r}_{\xi^k} + \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\partial A_{jk}}{\partial g_{mn}} (\underline{r}_{\xi^m} \cdot \underline{r}_{\xi^j \xi^n} + \underline{r}_{\xi^n} \cdot \underline{r}_{\xi^j \xi^m}) \underline{r}_{\xi^k} \tag{9}$$

for a weight function $w(\underline{\xi})$ in Eq. (1).

B. Two-Dimensional Examples

In two dimensions, the generation system (7) becomes (with $\xi^1 = \xi$ and $\xi^2 = \eta$)

$$\frac{\partial F}{\partial g_{11}} \underline{r}_{\xi\xi} + \frac{\partial F}{\partial g_{22}} \underline{r}_{\eta\eta} + 2 \frac{\partial F}{\partial g_{12}} \underline{r}_{\xi\eta}$$

$$+ \frac{\partial F'}{\partial g_{11}} (\underline{\nabla} w \cdot \underline{r}_\xi) \underline{r}_\xi + \frac{\partial F'}{\partial g_{22}} (\underline{\nabla} w \cdot \underline{r}_\eta) \underline{r}_\eta$$

$$+ \frac{\partial F'}{\partial g_{12}} [(\underline{\nabla} w \cdot \underline{r}_\xi) \underline{r}_\eta + (\underline{\nabla} w \cdot \underline{r}_\eta) \underline{r}_\xi]$$

$$+ \underline{r}_\xi \{\underline{r}_\xi \cdot [2 \frac{\partial^2 F}{\partial g_{11}^2} \underline{r}_{\xi\xi} + 4 \frac{\partial^2 F}{\partial g_{11} \partial g_{12}} \underline{r}_{\xi\eta}$$

$$+ (\frac{\partial^2 F}{\partial g_{12}^2} + \frac{\partial^2 F}{\partial g_{12} \partial g_{21}}) \underline{r}_{\eta\eta}]$$

$$+ \underline{r}_\eta \cdot [2 \frac{\partial^2 F}{\partial g_{11} \partial g_{12}} \underline{r}_{\xi\xi} + 2 \frac{\partial^2 F}{\partial g_{22} \partial g_{12}} \underline{r}_{\eta\eta}$$

$$+ (2 \frac{\partial^2 F}{\partial g_{11} \partial g_{22}} + \frac{\partial^2 F}{\partial g_{12}^2} + \frac{\partial^2 F}{\partial g_{12} \partial g_{21}}) \underline{r}_{\xi\eta}]\}$$

$$+ \underline{r}_\eta \{[(\frac{\partial^2 F}{\partial g_{12}^2} + \frac{\partial^2 F}{\partial g_{12} \partial g_{21}}) \underline{r}_{\xi\xi}$$

$$+ 2 \frac{\partial^2 F}{\partial g_{22}^2} \underline{r}_{\eta\eta} + 4 \frac{\partial^2 F}{\partial g_{22} \partial g_{12}} \underline{r}_{\xi\eta}] \cdot \underline{r}_\eta$$

$$+ \underline{r}_\xi \cdot [2 \frac{\partial^2 F}{\partial g_{11} \partial g_{12}} \underline{r}_{\xi\xi} + 2 \frac{\partial^2 F}{\partial g_{22} \partial g_{12}} \underline{r}_{\eta\eta}$$

$$+ (2 \frac{\partial^2 F}{\partial g_{11} \partial g_{22}} + \frac{\partial^2 F}{\partial g_{12}^2} + \frac{\partial^2 F}{\partial g_{12} \partial g_{21}}) \underline{r}_{\xi\eta}]\} - \frac{1}{2} F' \underline{\nabla} w = 0$$

If the weight function depends on $\underline{\xi}$, rather than on x, the terms $\underline{\nabla} w \cdot \underline{r}_\xi$ and $\underline{\nabla} w \cdot \underline{r}_\eta$ in Eq. (10) become w_ξ and w_η, respectively, and the last term, -- 1/2 $F' \underline{\nabla} w$, vanishes.

As an example, consider F_w from Eq. (XI-71). Then we have

$$\frac{\partial F}{\partial g_{11}} = w^2 g_{22}, \quad \frac{\partial F}{\partial g_{22}} = w^2 g_{11}, \quad \frac{\partial F}{\partial g_{12}} = -w^2 g_{21}, \quad \frac{\partial F}{\partial g_{21}} = -w^2 g_{12}$$

$$\frac{\partial A_{11}}{\partial g_{11}} = 0, \quad \frac{\partial A_{22}}{\partial g_{22}} = 2w^2, \quad \frac{\partial A_{11}}{\partial g_{12}} = 0$$

$$\frac{\partial A_{22}}{\partial g_{11}} = 2w^2, \quad \frac{\partial A_{22}}{\partial g_{22}} = 0, \quad \frac{\partial A_{22}}{\partial g_{12}} = 0$$

$$\frac{\partial A_{12}}{\partial g_{12}} = -w^2, \qquad \frac{\partial A_{12}}{\partial g_{11}} = \frac{\partial A_{12}}{\partial g_{22}} = 0$$

$$\frac{\partial F'}{\partial g_{11}} = 2wg_{22}, \qquad \frac{\partial F'}{\partial g_{22}} = 2wg_{11}, \qquad \frac{\partial F'}{\partial g_{12}} = \frac{\partial F'}{\partial g_{21}} = -2wg_{12}$$

Then the generation system based on concentration by Eq. (7) is

$$\begin{aligned}
&2w^2[g_{22}\mathbf{r}_{\xi\xi} + g_{11}\mathbf{r}_{\eta\eta} - 2g_{12}\mathbf{r}_{\xi\eta} - (\mathbf{r}_\eta \cdot \mathbf{r}_{\xi\xi})\mathbf{r}_\eta \\
&\quad - (\mathbf{r}_\xi \cdot \mathbf{r}_{\eta\eta})\mathbf{r}_\xi + (\mathbf{r}_\eta \cdot \mathbf{r}_{\xi\eta})\mathbf{r}_\xi + (\mathbf{r}_\xi \cdot \mathbf{r}_{\xi\eta})\mathbf{r}_\eta] \\
&\quad + 4w\{g_{22}(\nabla w \cdot \mathbf{r}_\xi)\mathbf{r}_\xi + g_{11}(\nabla w \cdot \mathbf{r}_\eta)\mathbf{r}_\eta \\
&\quad - g_{12}[(\nabla w \cdot \mathbf{r}_\xi)\mathbf{r}_\eta + (\nabla w \cdot \mathbf{r}_\eta)\mathbf{r}_\xi]\} - 2wg\,\nabla w = 0
\end{aligned} \tag{11}$$

With F taken to be a measure of orthogonality, i.e., F_o from Eq. (XI-70), we have,

$$\frac{\partial F}{\partial g_{11}} = \frac{(g_{11} - g_{22})g_{22} - 2g_{12}^2}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{22}} = \frac{(g_{22} - g_{11})g_{11} - 2g_{12}^2}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{12}} = \frac{(g_{11} + g_{22})g_{21}}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{21}} = \frac{(g_{11} + g_{22})g_{12}}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

The generation system based only on orthogonality then is

$$2[2g_{12}r_{\xi\eta} + (r_\xi \cdot r_{\eta\eta})r_\xi + (r_\eta \cdot r_{\xi\xi})r_\eta + (r_\xi \cdot r_{\xi\eta})r_\eta + (r_\eta \cdot r_{\xi\eta})r_\xi] = 0 \qquad (12)$$

Finally, for the smoothness integral, Eq, (XI-69), the derivatives needed are

$$\frac{\partial F}{\partial g_{11}} = \frac{(g_{11} - g_{22})g_{22} - 2g_{12}^2}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{22}} = \frac{(g_{22} - g_{11})g_{11} - 2g_{12}^2}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{12}} = \frac{(g_{11} + g_{22})g_{21}}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial F}{\partial g_{21}} = \frac{(g_{11} + g_{22})g_{12}}{2(g_{11}g_{22} - g_{12}^2)^{3/2}}$$

$$\frac{\partial A_{11}}{\partial g_{11}} = \frac{-g_{11}g_{22} + 4g_{12}^2 + 3g_{22}^2}{2(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{22}$$

$$\frac{\partial A_{11}}{\partial g_{22}} = \frac{-g_{22}^2 g_{11} + 8g_{22}g_{12}^2 - g_{11}^2 g_{22} + 2g_{12}^2 g_{11}}{2(g_{11}g_{22} - g_{12}^2)^{5/2}}$$

$$\frac{\partial A_{11}}{\partial g_{12}} = -\frac{2g_{11}g_{22} + g_{12}^2}{(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{12} = \frac{\partial A_{11}}{\partial g_{21}}$$

$$\frac{\partial A_{22}}{\partial g_{11}} = \frac{-g_{11}^2 g_{22} + 8g_{12}^2 g_{11} - g_{22}^2 g_{11} + 2g_{12}^2 g_{22}}{2(g_{11}g_{22} - g_{12}^2)^{5/2}}$$

$$\frac{\partial A_{22}}{\partial g_{22}} = \frac{-g_{11}g_{22} + 4g_{12}^2 + 3g_{11}^2}{2(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{11}$$

$$\frac{\partial A_{22}}{\partial g_{12}} = -\frac{2g_{11}g_{22} + g_{12}^2}{(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{12} = \frac{\partial A_{22}}{\partial g_{21}}$$

$$\frac{\partial A_{12}}{\partial g_{11}} = -\frac{g_{11}g_{22} + 2g_{12}^2 + 3g_{22}^2}{2(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{12}$$

$$\frac{\partial A_{12}}{\partial g_{22}} = -\frac{g_{11}g_{22} + 2g_{12}^2 + 3g_{11}^2}{2(g_{11}g_{22} - g_{12}^2)^{5/2}} \; g_{12}$$

$$\frac{\partial A_{12}}{\partial g_{12}} = \frac{g_{11}^2 g_{22} + 2g_{11}g_{12}^2 + g_{11}g_{22}^2 + 2g_{22}g_{12}^2}{2(g_{11}g_{22} - g_{12}^2)^{5/2}} = \frac{\partial A_{12}}{\partial g_{21}}$$

The complete generation system is then obtained as the linear combination of the concentration system, Eq. (11), the orthogonality system, Eq. (12), and the smoothness system which is formed by substituting the above relations into the general equations (7). The three-dimensional case follows in an analogous fashion.

2. Variational Principle in Physical Space

With the variational problem formulated in the physical space, consider the integral

$$I = \int F[\bar{g}, w(\underline{\xi})]d\underline{x} \tag{13}$$

where $\bar{g}$ is the contravariant metric tensor, i.e., with elements g^{ij} from Eq. (III-37), and the weight function is a function of $\underline{\xi}$.

A. Grid Generation System

Then for the Euler equations, we have

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial F}{\partial (\xi^i)_{x_j}} - \frac{\partial F}{\partial \xi^i} = 0 \quad (i = 1,2,3) \tag{14}$$

Now,

$$(\xi^i)_{x_j} = (\underline{\nabla}\xi^i)_j = (\underline{a}^i)_j$$

and F depends on $(\xi^i)_{x_j}$ only through $\bar{g}$. Then

$$\frac{\partial F}{\partial (\xi^i)_{x_j}} = \frac{\partial F}{\partial (\underline{a}^i)_j} = \frac{\partial F}{\partial \underline{a}^i} \frac{\partial \underline{a}^i}{\partial (\underline{a}^i)_j} = \frac{\partial F}{\partial \underline{a}^i} \underline{e}_j$$

Also, since F depends on $\underline{a}^i$ only through g^{ik} (k = 1,2,3) we have

$$\frac{\partial F}{\partial \underline{a}^i} \underline{e}_j = \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g^{lk}} \frac{\partial g^{lk}}{\partial \underline{a}^i} \underline{e}_j$$

$$= \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g^{lk}} \frac{\partial (\underline{a}^l \cdot \underline{a}^k)}{\partial \underline{a}^i} \underline{e}_j$$

$$= \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial F}{\partial g^{lk}} (\delta_{li}\, \underline{a}^k \cdot \underline{e}_j + \delta_{ki}\, \underline{a}^l \cdot \underline{e}_j)$$

Therefore,

$$\frac{\partial F}{\partial(\xi^i)_{x_j}} = \sum_{k=1}^{3} \left(\frac{\partial F}{\partial g^{ki}} + \frac{\partial F}{\partial g^{ik}}\right) \underline{a}^k \cdot \underline{e}_j$$

Also, since F depends on $\underline{\xi}$ only through the weight function, we have

$$\frac{\partial F}{\partial \xi^i} = \frac{\partial F}{\partial w} \frac{\partial w}{\partial \xi^i}$$

Then the Euler equations can be written

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\partial}{\partial x_j} \left[\left(\frac{\partial F}{\partial g^{ik}} + \frac{\partial F}{\partial g^{ki}}\right) \underline{a}^k \cdot \underline{e}_j\right] - \frac{\partial F}{\partial w} \frac{\partial w}{\partial \xi^i} = 0$$

$$(i = 1,2,3)$$

or

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \left[\left(\frac{\partial F}{\partial g^{ik}} + \frac{\partial F}{\partial g^{ki}}\right) \underline{e}_j \cdot (\underline{a}^k)_{x_j}\right.$$

$$\left. + (\underline{a}^k \cdot \underline{e}_j) \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial g^{ik}} + \frac{\partial F}{\partial g^{ki}}\right)\right] - \frac{\partial F}{\partial w} \frac{\partial w}{\partial \xi^i} = 0 \quad (i = 1,2,3)$$

Now

$$\underline{e}_j \cdot (\underline{a}^k)_{x_j} = \underline{e}_j \cdot (\underline{\nabla} \xi^k)_{x_j} = (\xi^k)_{x_j x_j}$$

and

$$\underline{a}^k \cdot \underline{e}_j = (\xi^k)_{x_j}$$

Then

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\left[\left(\frac{\partial F}{\partial g^{ik}}+\frac{\partial F}{\partial g^{ki}}\right)\xi^k_{x_j x_j}+\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial g^{ik}}+\frac{\partial F}{\partial g^{ki}}\right)\xi^k_{x_j}\right]$$

$$-\frac{\partial F}{\partial w}\frac{\partial w}{\partial \xi^i}=0$$

or,

$$\sum_{k=1}^{3}\left(\frac{\partial F}{\partial g^{ik}}+\frac{\partial F}{\partial g^{ki}}\right)\nabla^2\xi^k+\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial g^{ik}}+\frac{\partial F}{\partial g^{ki}}\right)\xi^k_{x_j}$$

$$-\frac{\partial F}{\partial w}\frac{\partial w}{\partial \xi^i}=0 \qquad (i=1,2,3)$$

Now

$$\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial g^{ik}}\right)\xi^k_{x_j}=\underline{\nabla}\left(\frac{\partial F}{\partial g^{ik}}\right)\cdot\underline{\nabla}\xi^k$$

and

$$\underline{\nabla}\left(\frac{\partial F}{\partial g^{ik}}\right)=\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 F}{\partial g^{mn}\partial g^{ik}}\underline{\nabla}g^{mn}+\left[\frac{\partial}{\partial g^{ik}}\left(\frac{\partial F}{\partial w}\right)\right]\sum_{l=1}^{3}\frac{\partial w}{\partial \xi^l}\underline{\nabla}\xi^l$$

$$=\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial^2 F}{\partial g^{mn}\partial g^{ik}}\left[(\underline{\nabla}\xi^m\cdot\underline{\nabla})\underline{\nabla}\xi^n+(\underline{\nabla}\xi^n\cdot\underline{\nabla})\underline{\nabla}\xi^m\right]$$

Then the generation system is, with $\partial F/\partial w$ written as F',

$$\sum_{k=1}^{3}\{A_{ik}\nabla^2\xi^k+A'_{ik}\sum_{l=1}^{3}\frac{\partial w}{\partial \xi^l}\underline{\nabla}\xi^l\cdot\underline{\nabla}\xi^k$$

$$+\sum_{m=1}^{3}\sum_{n=1}^{3}\frac{\partial A_{ik}}{\partial g^{mn}}[\underline{\nabla}(\underline{\nabla}\xi^m\cdot\underline{\nabla}\xi^n)]\cdot\underline{\nabla}\xi^k\} \qquad (15)$$

$$-F'\frac{\partial w}{\partial \xi^i}=0 \qquad (i=1,2,3)$$

where

$$A_{ik} = \frac{\partial F}{\partial g^{ik}} + \frac{\partial F}{\partial g^{ki}} \tag{16}$$

This can also be written as

$$\sum_{k=1}^{3} A_{ik} \nabla^2 \xi^k = S_i \qquad (i = 1,2,3) \tag{17}$$

$$\begin{aligned} S_i = - \sum_{k=1}^{3} \Big\{ \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\partial A_{ik}}{\partial g^{mn}} \left[\underline{\nabla}(\underline{\nabla}\xi^m \cdot \underline{\nabla}\xi^n)\right] \cdot \underline{\nabla}\xi^k \\ - A'_{ik} \sum_{l=1}^{3} \frac{\partial w}{\partial \xi^l} \underline{\nabla}\xi^l \cdot \underline{\nabla}\xi^k \Big\} + F' \frac{\partial w}{\partial \xi^i} \end{aligned} \tag{18}$$

Then

$$\nabla^2 \xi^i = \frac{1}{\det|\underline{A}|} \sum_{k=1}^{3} C_{ik} S_k \tag{19}$$

where C_{ik} is the signed cofactor of A_{ki}.

If the weight function in the integral (13) is a function of $\underset{\sim}{x}$, rather than $\underset{\sim}{\xi}$, then $\partial F / \partial \xi^i = 0$ in the Euler equation (14), and Eq. (15) is replaced by

$$\begin{aligned} \sum_{k=1}^{3} \Big\{ \left[A_{ik} \nabla^2 \xi^k + A'_{ik} \underline{\nabla} w \cdot \underline{\nabla}\xi^k\right] \\ + \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\partial A_{ik}}{\partial g^{mn}} \left[\underline{\nabla}(\underline{\nabla}\xi^m \cdot \underline{\nabla}\xi^n)\right] \cdot \underline{\nabla}\xi^k \Big\} = 0 \end{aligned} \tag{20}$$

In this case S_i of Eq. (18) are redefined as

$$S_i = - \sum_{k=1}^{3} \Big\{ \sum_{m=1}^{3} \sum_{n=1}^{3} \frac{\partial A_{ik}}{\partial g^{mn}} \left[\underline{\nabla}(\underline{\nabla}\xi^m \cdot \underline{\nabla}\xi^n)\right] \cdot \underline{\nabla}\xi^k - A'_{ik} \underline{\nabla} w \cdot \underline{\nabla}\xi^k \Big\} \tag{21}$$